1	2	3	4	5	6	7	8	9	10
b	a	d	d	c	c	c	c	c	b
11	12	13	14	15	16	17	18	19	20
d	e	b	a	c	c	d	b	b	a
21	22	23	24	25					
e	c	d	c	c					

1. 假設 $a_1, a_2, \cdots, a_{2016}$ 是整數，且 $\sum_{k=1}^{2016} a_k^2 = a_1^2 + a_2^2 + \cdots + a_{2016}^2 = 2016$。請問 $a_1, a_2, \cdots, a_{2016}$ 中最多可以有幾個零? Suppose that $a_1, a_2, \cdots, a_{2016}$ are integers, and $\sum_{k=1}^{2016} a_k^2 = a_1^2 + a_2^2 + \cdots + a_{2016}^2 = 2016$. Please find the maximum number of zeros in $a_1, a_2, \cdots, a_{2016}$.

Solution: Since 2016 is not a square number and $\sqrt{2016} \approx 44.0$. Therefore, we have $2016 = 44^2 + 80$. And 80 is not a square number, either. In addition, we have $80 = 8^2 + 16 = 8^2 + 4^2$. Therefore, we can set $a_1 = 44$, $a_2 = 8$, $a_3 = 4$, $a_k = 0$ for $k > 3$. Therefore we can have $2016 - 3 = 2013$ at most.

(a) 2014
(b) 2013
(c) 2012
(d) 2011
(e) 以上皆非 None of the above

2. 2016 的所有正因數總和是多少？Find the sum of all the positive factors of 2016.

Solution: $2016 = 2^5 \cdot 3^2 \cdot 7$. The sum is $(1+2+4+8+16+32)(1+3+9)(1+7) = 6552$

(a) 6552
(b) 3224
(c) 5208
(d) 78624
(e) 以上皆非 None of the above

3. 有一個三位數 abc，它的三個數字均不同。將它的百位數移到個位數變成 bca，再做一次同樣的動作變成 cab。請問此三個數中 (abc, bca, cab) 最大的與最小的差不可能是下面的哪一個數字？We have a three-digit number abc. The three digits are different from each other. We move the first digit of abc to the last one, i.e., the number becomes bca. Repeat the same procedure we have cab. Among all three numbers (abc, bca, and cab), which of the following number cannot be the difference between the largest and smallest one?

Solution: Suppose $a > b > c$. $abc - cab$ is $100a + 10b + c - (100c + 10a + b) = 90a + 9b - 99c = 9(10a + b - 11c)$. Solve $10a + b = 21 + 11c, 32 + 11c, 43 + 11c, 50 + 11c$. It is easy to check we cannot have $a > b > c$ when $10a + b = 50 + 11c$ ($c = 0, 1, 2, 3, 4$, $ab = 50 + 11c = 50, 61, 72, 83, 94$) The other case is $a > c > b$. $abc - bca$ is $100a + 10b + c - (100b + 10c + a) = 99a - 90b - 9c = 9(11a - 10b - c)$. Similarly, we cannot have $a > c > b$ when $10b + c = 11a - 50$ and $a = 5, 6, 7, 8, 9$.

(a) 189
(b) 288
(c) 387
(d) 450
(e) 以上皆非 None of the above

4. 試求 $4^x - 3 \cdot 2^{x+15} - 2^{32} = 0$ 的根. Find the root of $4^x - 3 \cdot 2^{x+15} - 2^{32} = 0$.

Solution: Let $t = 2^x$, $4^x - 3 \cdot 2^x - 2^{32} = t^2 - 3 \cdot 2^{15} t - 2^{32} = (t - 4 \cdot 2^{15})(t + 2^{15}) = 0$. Therefore, $x = 17$.

(a) 14
(b) 15
(c) 16
(d) 17
(e) 以上皆非 None of the above

5. 直線 $L : 3x + 4y - k = 0$。如果我們用 $d(P, L)$ 表示 P 到直線 L 的距離，試求 k 使得 $A(1, 3)$、$B(2, 7)$、$C(3, 2)$ 到直線 L 距離平方和 $d(A, L)^2 + d(B, L)^2 + d(C, L)^2$ 最小。Please find k such that the sum of the square of the distance between the line $L : 3x + 4y - k = 0$ and three points $A(1, 3)$, $B(2, 7)$, and $C(3, 2)$, i.e., to minimize of the function $d(A, L)^2 + d(B, L)^2 + d(C, L)^2$. The symbol $d(P, L)$ stands for the distance between a point P and a line L.

Solution: The sum of the square of ditances $d(A, L)^2 + d(B, L)^2 + d(C, L)^2 = \frac{1}{25}\left((15 - k)^2 + (34 - k)^2 + (17 - k)^2\right)$. When $k = \frac{15+17+34}{3} = 22$, the function reaches its minimum.

(a) 20
(b) 21
(c) 22
(d) 23
(e) 以上皆非 None of the above

6. 將一杯 75°C 的水放在恆溫 30°C 的環境中。水溫每隔 10 分鐘就會下降水溫與室溫差距的一半。請問幾分鐘後可以下降到 35°C？The temperature of water in a cup is 75°C. We put the cup in a enviorment with temperature 30°C. The temperature of water decay half of the temperature difference between the water and the enviorment every ten minutes. How many minutes does it take till the temperature of water down to 35°C?

Solution: Suppose we need n minutes. Since $75 - 30 = 45$ and $35 - 30 = 5$, we need $45 \cdot 2^{-\frac{n}{10}} = 5 \Rightarrow 20 = 2^{\frac{n}{10}}$. Therefore, $n = 10 \log_2 9 = 20 \log_2 3$

(a) $\log_2 90$
(b) $10 \log_2 3$
(c) $20 \log_2 3$
(d) $10 \log_2 \frac{75}{35}$
(e) 以上皆非 None of the above

7. 多項式 $f(x) = x^7 - 6x^6 - 5x^5 - 14x^4 + 6x^3 + x^2 - 13x + 1$。試求 $f(7)$。Suppose a polynomial $f(x) = x^7 - 6x^6 - 5x^5 - 14x^4 + 6x^3 + x^2 - 13x + 1$. Please compute $f(7)$.

Solution: If $x = 7$, we have $f(7) = 2017$ by synthetic division.

(a) 2015
(b) 2016
(c) 2017
(d) 2018
(e) 以上皆非 None of the above

8. 有一學生用銅板猜考題的答案。這個考題有五個選項但只有一個正確答案。他對每個選項丟銅板決定。如果是正面就選它，反面就不選。如果丟出來有超過一個選項的話，他再任意挑一個作答。請問他猜對的機率有多少？A student use a coin to guess the answer for a question. The question has five options and only one correct answer. He tosses up the coin for each option. The option is chosen if the outcome is head. If there are more than one chosen option, he randomly chooses one for the answer. What is the probability that he can get the correct answer?

Solution: We can compute the probability accroding to the number of chosen option.
The answer is $\frac{1}{2^5}\left(C^4_0 \cdot 1 + C^4_1 \cdot \frac{1}{2} + C^4_2 \cdot \frac{1}{3} + C^4_3 \cdot \frac{1}{4} + C^4_4 \cdot \frac{1}{5}\right) = 31/160$

(a) 1/5
(b) 7/32
(c) 31/160
(d) 33/160
(e) 以上皆非 None of the above

9. 三角形 ABC 中，$\angle A = 30°$，$\overline{AB} = 6$，$\overline{BC} = 3\sqrt{2}$。如果我們收集所有可能出現的 $\angle B$，$\angle C$ 在集合 S 中，請問集合 S 中不會出現下面的哪一個角度？In triangle ABC, $\angle A = 30°$, $\overline{AB} = 6$, $\overline{BC} = 3\sqrt{2}$. If we collect all the possible angles of $\angle B$ and $\angle C$ in a set S, which of the following is not in the set S?

Solution: There are two possibilities. One is ($\angle B = 15°$ and $\angle C = 135°$). The other is ($\angle B = 105°$ and $\angle C = 45°$). Therefore, the set S does not contain 75°.

(a) 15°
(b) 45°
(c) 75°
(d) 105°
(e) 以上皆非 None of the above

10. 數列 a_k 滿足 $a_{k+1} = 2^k \times a_k$，且 k 是正整數且 $a_1 = 1$。請問 $\log_2 a_n$ 的一般式是多少? A sequence a_k satisfies $a_{k+1} = 2^k \times a_k$, where k is an positive integer and $a_1 = 1$. What is the general formula of $\log_2 a_n$?

Solution: $a_{k+1} = 2^k a_k \Rightarrow \log_2 a_{k+1} - \log_2 a_k = k \Rightarrow \sum\limits_{k=1}^{n-1} \log_2 a_{k+1} - \log_2 a_k = \log_2 a_n - \log_2 a_1 = \sum_{k=1}^{n-1} k = \frac{n(n-1)}{2}$.

(a) $n - 1$
(b) $\frac{n(n-1)}{2}$
(c) $n(n-1)$
(d) $2^{n-1} - 1$
(e) 以上皆非 None of the above

11. z 是一個虛部非零的複數，且 $z+\frac{1}{z}=\sqrt{3}$，試求 $z^{2016}+z^{-2016}$。Suppose z is a complex number and the imagnary part of z is not zero. If $z+\frac{1}{z}=\sqrt{3}$, please compute $z^{2016}+z^{-2016}$.

Solution: $z+\frac{1}{z}=\sqrt{3} \Rightarrow z^2-\sqrt{3}z+1=0$. Therefore, $z=\frac{\sqrt{3}\pm i}{2}=\cos\theta+i\sin\theta$, where $\theta=\pm\frac{\pi}{6}$. $z^{2016}+z^{-2016}=2\cos 2016\theta=2\cos(0)=2$.

(a) -2
(b) $-\sqrt{3}$
(c) 1
(d) 2
(e) 以上皆非 None of the above

12. 一物體在平面的運動軌跡為 $(x(t),y(t))=(5\cos t+3\sin t,4\sin t)$。請問該物體離原點最遠的距離是多少？The orbit of an object is $(x(t),y(t))=(5\cos t+3\sin t,4\sin t)$. What is the longest distance between the object and the origin $(0,0)$?

Solution: $x^2+y^2=25\cos^2 t+25\sin^2 t+30\sin t\cos t=25+15\sin 2t\leq 40$. So the logest distance is $\sqrt{40}$. None of the following will be the answer.

(a) 5
(b) 6
(c) 7
(d) 8
(e) 以上皆非 None of the above

13. 將 ABABCAB 這七個英文字重新排列，不出現任何 AB 子字串的排法有幾種？If we re-arrange the characters in the word ABABCAB, how many possible arrangments that the re-arranged word does not contain any sub-word AB? For example, B<u>AB</u>CBAA is not allowed since the underline part.

Solution: BBBAAAC,BBBAACA,BBBACAA,BBBCAAA;
BBAAACB,BBAACBA,BBACBAA,BBCBAAA;
BAAACBB,BAACBBA,BACBBAA,BCBBAAA;
AAACBBB,AACBBBA,ACBBBAA,CBBBAAA. 16 possibilities.

(a) 20
(b) 16
(c) 12
(d) 26
(e) 以上皆非 None of the above

14. 請找出以 $y=\sqrt{x}$ 及 $y=x$ 之間的區域繞 x 軸旋轉一周所形成的旋轉體體積。Find the volume of solid obtained by rotating the region bounded by $y=\sqrt{x}$, $y=x$ about the x-axis.

Solution: The intersections of $y=\sqrt{x}$ and $y=x$ are $(0,0)$ and $(1,1)$. Therefore, the volume is $\int_0^1 \pi\left((\sqrt{x})^2-x^2\right)dx=\frac{\pi}{6}$.

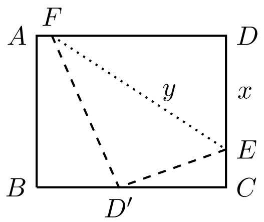
(a) $\frac{\pi}{6}$
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$
(d) $\frac{\pi}{2}$
(e) 以上皆非 None of the above

15. 計算 $\lim\limits_{x\to 2}\dfrac{\sqrt{7+x}-3}{1-\sqrt{3-x}}$。Evaluate $\lim\limits_{x\to 2}\dfrac{\sqrt{7+x}-3}{1-\sqrt{3-x}}$.

Solution: $\lim\limits_{x\to 2}\frac{\sqrt{7+x}-3}{1-\sqrt{3-x}}=\lim\limits_{x\to 2}\frac{(1+\sqrt{3-x})(7+x-9)}{(\sqrt{7+x}+3)(1-3+x)}=\frac{1}{3}$

(a) $\frac{1}{6}$
(b) $\frac{1}{4}$
(c) $\frac{1}{3}$
(d) $\frac{1}{2}$
(e) 以上皆非 None of the above

16. 有一張長 $\overline{AD}=20$ 公分、寬 $\overline{AB}=16$ 公分的紙，如下圖折起右上角 D 貼於底邊 D' 上。請問當 $\overline{DE}=x$ 是多少時，折下去的折痕 $\overline{EF}=y$ 是最短的? The paper is $\overline{AD}=20$cm by $\overline{AB}=16$cm. The upper-right corner D is folded over to D' at the bottom edge, see the following figure. How would you choose $\overline{DE}=x$ in order to minimize the length of the fold ($\overline{EF}=y$)?



Solution: $\overline{CE}=16-x$, $\overline{D'E}=x$, $\overline{D'C}=\sqrt{x^2-(16-x)^2}=\sqrt{32x-16^2}$, $\overline{D'F}=20\cdot\frac{\overline{D'E}}{\overline{D'C}}=\frac{16x}{\sqrt{32x-16^2}}$. If we consider $y^2=\overline{D'E}^2+\overline{D'F}^2=x^2+\frac{8x^2}{x-8}$. By taking the derivative, we have $2x+\frac{16x}{x-8}-\frac{8x^2}{(x-8)^2}=\frac{2x^2(x-12)}{(x-8)^2}$. Therefore, when $x=12$, the length of fold y reaches its minimum.

(a) 8
(b) 10
(c) 12
(d) 14
(e) 以上皆非 None of the above

17. 若 $y=f(x)=x^2+kx+(k^2-5k+3)$ 與 x 軸有交點，試求 k 的最大值。The function $y=f(x)=x^2+kx+(k^2-5k+3)$ and x-axis has at least one intersection. What is the maximum value of k?

Solution: $k^2-4(k^2-5k+3)\geq 0 \Rightarrow 3k^2-20k+12=(3k-2)(k-6)\leq 0 \Rightarrow \frac{2}{3}\leq k\leq 6$. Therefore, the maximum of k is 6.

(a) $\frac{2}{3}$
(b) 0
(c) 1
(d) 6
(e) 以上皆非 None of the above

18. 請問 20^{16} 的每一位數字的總和是多少？Please compute the sum of every digit of 20^{16}.

Solution: $20^{16}=2^{16}\times 10^{16}=655360000000000000000$. Therefore, the sum is $6+5+5+3+6=25$. In addition, the remainder of sum of digits divided by 9 is 7.

(a) 15
(b) 25
(c) 1024
(d) 2^{16}
(e) 以上皆非 None of the above

19. D 在三角形 ABC 的 $\overline{BC}$ 線段上，且 $\overline{BD}:\overline{CD}=3:5$。$E$ 在 $\overline{AD}$ 上且 $\overline{AE}:\overline{AD}=3:5$，若向量 $\overrightarrow{AE}$ 可以表示成 $\alpha\overrightarrow{AB}+\beta\overrightarrow{AC}$，試求 $\alpha+\beta$。The point D is located on the $\overline{BC}$ of triangle ABC and the point E is on $\overline{AD}$. Suppose $\overline{BD}:\overline{CD}=3:5$, and $\overline{AE}:\overline{AD}=3:5$. If the vector $\overrightarrow{AE}=\alpha\overrightarrow{AB}+\beta\overrightarrow{AC}$, evaluate $\alpha+\beta$.

Solution: $\overrightarrow{AE}=\frac{3}{5}\overrightarrow{AD}=\frac{3}{5}\left(\frac{5}{8}\overrightarrow{AB}+\frac{3}{8}\overrightarrow{AC}\right)=\frac{15}{40}\overrightarrow{AB}+\frac{9}{40}\overrightarrow{AC}$. Therefore $\alpha+\beta=\frac{15}{40}+\frac{9}{40}=\frac{3}{5}$.

(a) 1
(b) $\frac{3}{5}$
(c) $\frac{17}{32}$
(d) $\frac{5}{8}$
(e) 以上皆非 None of the above

20. $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$，若 $A^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$，下列敘述何者不一定正確？$A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $A^2=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, which of the following statements is <u>NOT</u> always true?

Solution: $A^2=\begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}$. Therefore, the solution for A is either $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or the two conditions, $a+d=0$ and $a^2+bc=d^2+bc=0$, are satisfied.

(a) $b=c=0$
(b) $a+d=0$
(c) $a^2+bc=0$
(d) $ad-bc=0$
(e) 以上皆非 None of the above

21. 有三位法師海濤、裝圓、不嚴。海濤法師不論問什麼他都說『假的』或『不是』。裝圓法師則是聽到上一位回答什麼他就說什麼，如果沒有上一個回答，也就是他是第一位，他就會照實回答。不嚴法師則有 1/5 的機率會說謊。請問，如果三位法師排成一列，但我們不知道哪一位法師在哪一個位置，但每種排列的可能性都一樣。當由左而右問他們，請問這題的答案是 (e) 嗎？假設三位都知道答案，但三位法師的回答都是『假的』或『不是』。請問這題的答案是 (e) 的可能性有多少？We have three guests A, B, C in the TV show. The answer from A is always FALSE or NO. If B hears the previous answer, he repeats the answer. However, if B is the first one, he gives the true answer. C tells a lie with probability 1/5. Today A, B, and C is in a row but we don't know the order. Suppose the probability of each order is equal. If we ask them: is the answer (e)? And they all answer NO. What is the probability that (e) is the answer? By the way, they all know the correct answer.

Solution: ABC or ACB: $\frac{1}{6} \cdot \frac{1}{5}$.
BAC or BCA: 0
CAB or CBA: $\frac{1}{6} \cdot \frac{1}{5}$. Therefore, the answer is $\frac{4}{30}$, i.e., (e) None of the above.

(a) $\frac{1}{30}$
(b) $\frac{1}{15}$
(c) $\frac{1}{10}$
(d) $\frac{8}{15}$
(e) 以上皆非 None of the above

22. x，y，z 都是正實數。已知 $x + \frac{y}{4} + \frac{z}{4} = 1$，試求 $\sqrt{x} + \sqrt{y} + \sqrt{z}$ 的最大值。Suppose that x, y, and z are positive real number. Find the maximum value for $\sqrt{x} + \sqrt{y} + \sqrt{z}$ under the condition $x + \frac{y}{4} + \frac{z}{4} = 1$.

Solution: By Cauchy inequality, we have $\left(\sqrt{x}^2 + \left(\frac{\sqrt{y}}{2}\right)^2 + \left(\frac{\sqrt{z}}{2}\right)^2\right)(1^2 + 2^2 + 2^2) \geq (\sqrt{x} + \sqrt{y} + \sqrt{z})^2$. Therefore, $3 \geq \sqrt{x} + \sqrt{y} + \sqrt{z}$

(a) 1
(b) 2
(c) 3
(d) 4
(e) 以上皆非 None of the above

23. 四面體 $ABCD$ 滿足 $\overline{AB} = \overline{CD} = 4$, $\overline{AC} = \overline{AD} = \overline{BC} = \overline{BD} = 3$。若 θ 是平面 ABD 及 ACD 的二面角，試求 $\sin\theta$？We have a tetrahedron $ABCD$, $\overline{AB} = \overline{CD} = 4$, $\overline{AC} = \overline{AD} = \overline{BC} = \overline{BD} = 3$. If θ is the dihedral angle of ABD and ACD, what is $\sin\theta$?

Solution: Let $A = (0, 0, 2)$, $B = (0, 0, -2)$, $C = (d, 2, 0)$, $D = (d, -2, 0)$. Since $\overline{AC} = 3$, we have $d^2 + 4 + 4 = 9 \Rightarrow d = 1$. We can compute the equations of two planes $ABD : 2x + y = 0$, $ACD : 2x + z = 2$. Therefore $\cos(\theta) = 4/5 \Rightarrow \sin(\theta) = 3/5$

(a) 1/2
(b) 1/3
(c) 3/4
(d) 3/5
(e) 以上皆非 None of the above

24. 班上有 30 個學生，他們班的數學平均成績是 70 分 (分數為 0 到 100 分)。A，B，C 三位老師各從班上抽了 10 名學生，並計算他們的平均數學成績各為 60、72 及 78 分。以下敘述何者不可能成真？(如果前四個選項都有可能，請選 (e)) We have 30 students in the class. The average score of math is 70 (possible score ranging from 0 to 100). Teachers A, B and C pick 10 students from the class to form a group and the average score of the 10 students in each group is 60, 72, and 78, respectively. Which of the following statements is not possible? (If options (a), (b), (c), (d) are possible, please choose (e))

Solution: A: B: 10 個 60 分，10 個 72 分及 10 個 78 分即可。C、D: 20 個 100 分，1 個 20 分，1 個 80 分，剩下 0 分。所以答案是 E

(a) A，B，C 所抽出來三組人中沒有任何交集。The groups chosen by A, B, and C do not have any intersection.
(b) 全班的數學成績變異數為 56。The variance of the math score of 30 students is 56.
(c) 再抽 10 個人的成績，抽出來 10 個人的平均分數是 10 分。If we pick 10 students again, it is possible that their average score is 10.
(d) 這個班級的中位數是 100 分。The medium score of the 30 students is 100.
(e) 以上皆可能 All of the above are possible.

25. 如果最後一題的題目是「在這次考試中，答案中出現最多的選項是哪一個？」在你看題目的選項之前，你就很確定答案只有一個。請問下面敘述何者錯誤？(如果前四個選項都正確，請選 (e)) If the last question is “What is the most option that you choose in this exam?”, and you are sure that you only has one answer before you check the options in the question. Which of the following option is wrong? (If options (a), (b), (c), (d) are true, please choose (e))

Solution: C: 錯誤。有可能前面 24 題各選項的數目是 9,5,5,3,2。其他正確

(a) 最高的選項數最後一定比次高的選項數多 2 以上。The difference between the highest counts of options and the second highest one is not less than 2.
(b) 最後 25 題全做完，最高的選項數一定在 7 次以上。The highest counts of options is not less than 7 including all the answers in the exam.
(c) 前面 24 題的答案選項中，一定沒有兩個選項出現同樣多次。The counts of each options are different from each other in the previous 24 answers.
(d) 前面 24 題的答案選項中，出現最多次的只有一個。The highest counts of options is only one.
(e) 以上皆正確 All of the above are correct.